Definitions [edit]

The Dirac delta function $\delta(x)$ can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) \simeq \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and which is also constrained to satisfy the identity[23]

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

This is merely a heuristic characterization. The Dirac delta is not a function in the traditional sense as no extended real number valued function defined on the real numbers has these properties.[24]

- δ(x) is called Dirac delta function or Unit Impulse function (because its total definite integral over real line is 1 and it was introduced by Paul Dirac).
- Because δ(x) provided a way to model point charge from a charge distribution (such as a uniformly charged rod); later the name "Distribution" was coined by Laurent Schwartz.
- So Dirac delta function is also called Dirac delta distribution.
- Distribution is other name for Generalized functions.

Terminologies

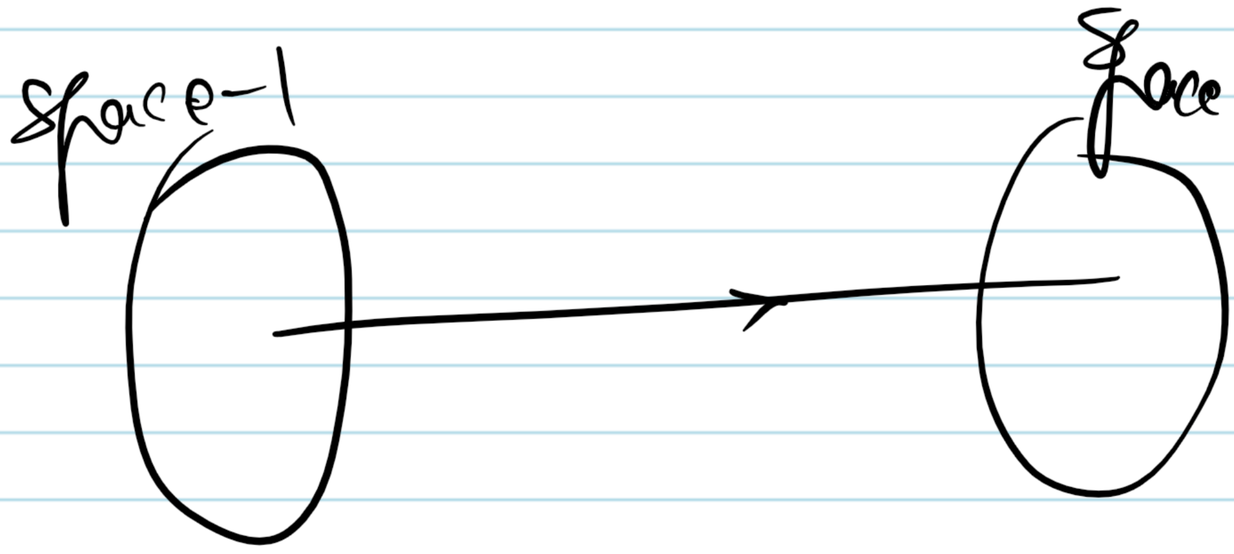
① Set { 1, 2, 3, a, b, c }

Spaces = set + underlying Structure.

→ { a, A, b, B, — — — — }

Terminologies

① Sets vs Spaces



Functionals?

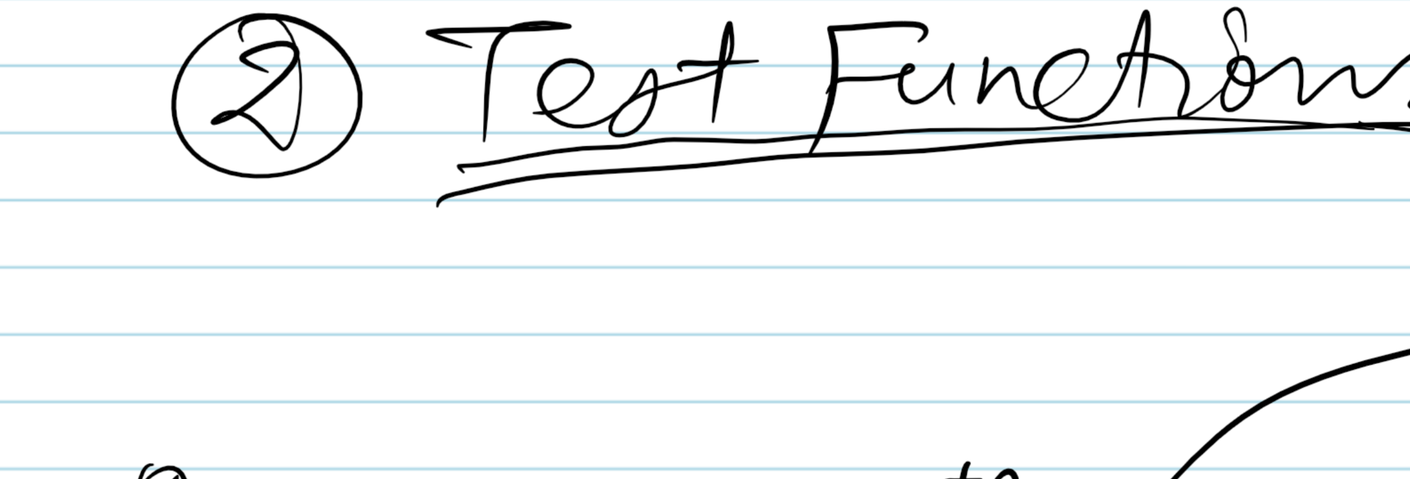
② Test Functions

① $\neq 0$

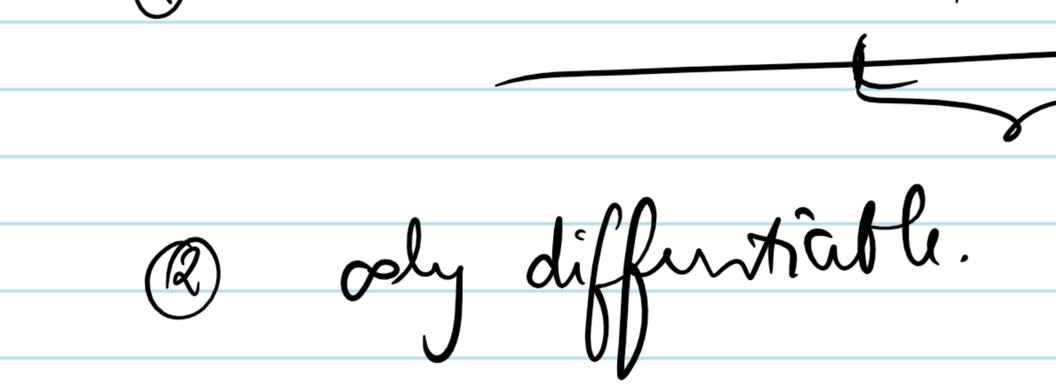
② only differentiable.

Compact Support

Support $\neq 0$



→ not a test function because it is not only differentiable

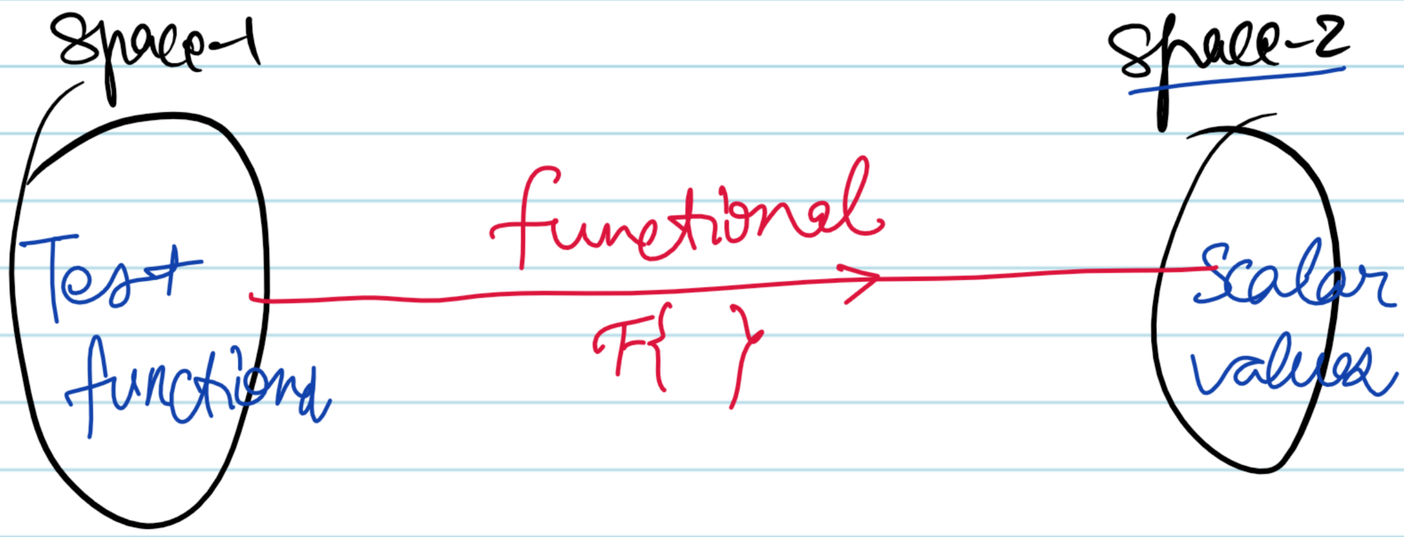


→ It is a test function.

$\mathbb{R} \to \mathbb{R}$

$e^{-\frac{1}{1-x^2}}$

(3) Functionals



$\Rightarrow \quad F\{ f(x) \} = \text{number}$

$$\begin{cases} I/p = f(x) / \text{test function} \\ O/p = \text{scalar number} \end{cases}$$

$f(x) = x^2$

$f : \mathbb{R} \to \mathbb{R}$

$I/p = \in \mathbb{R}$ ~~(x)~~ X

$O/p = \in \mathbb{R}$

function ✓

functional X

✓ $f(g(x))$

$R \to R$

$x = I/p \in \mathbb{R}$

* Functional ⇔

$I/p =$ ~~test~~ function" or vector

$O/p =$ scalar no.

→ Functionals **act** upon test functions.

Action?

eg. $F\{f(x)\} = \int_1^{10} f(x)\,dx$

eg. point evaluation $E_{x_0}\{f(x)\} = f(x_0)$

$\vec{V}$ → $\Phi_{\hat{u}}$ → projection along $\hat{u}$

$\Phi_{\hat{u}}(\vec{V}) = \vec{V} \cdot \hat{u}$

① f(x)

② action ?

functional

Functional

I/p = test function

o/p = scalar

action

---

① $F\{f+g\} = F\{f\} + F\{g\}$ ($S_1 = S_2 + S_3$)

② $F\{\lambda f\} = \lambda \times F\{f\}$ ($S_4 = \lambda \times S_5$)

linearity *

$F\{\ \}$

Linear Functional

continuous *

Generalized function / Distribution

e.g. $\delta(x)$

$\{\phi_k\}$

$\Rightarrow \{\phi_1, \phi_2, \phi_3 \ldots\} \longrightarrow$ converge to 0.

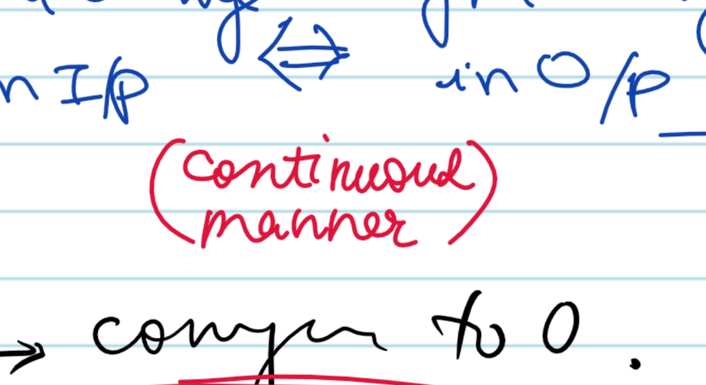
linear $F\{\ \}$

where $\lambda_i = F\{\phi_i\}$

$\{\lambda_1, \lambda_2, \lambda_3 \ldots\} \longrightarrow$ converge to 0.

$\{F[\phi_k]\}$

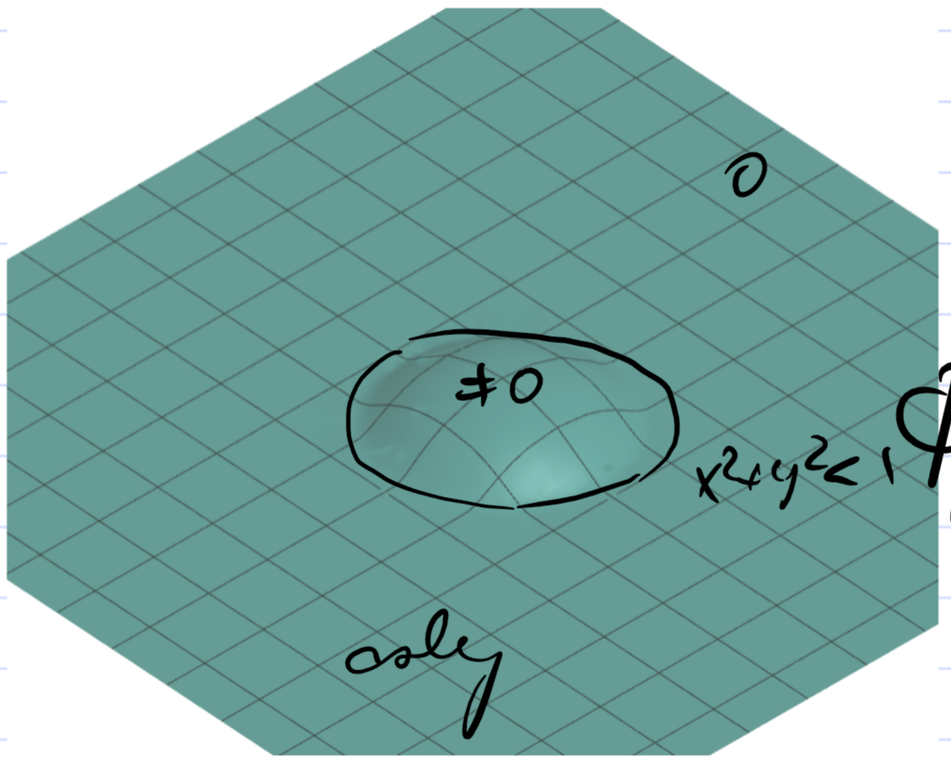
$F\{f(x)\} = \int_0^1 f(x)\,dx$ (= 10)



no abrupt change in the O/p for slight change in I/p

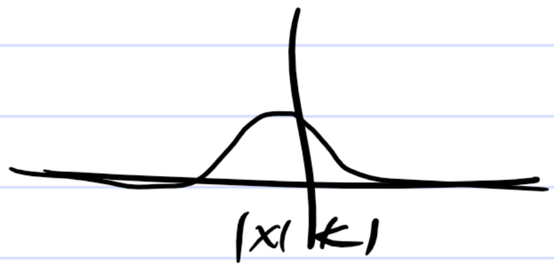
[slight change in I/p ⇔ slight change in O/p]

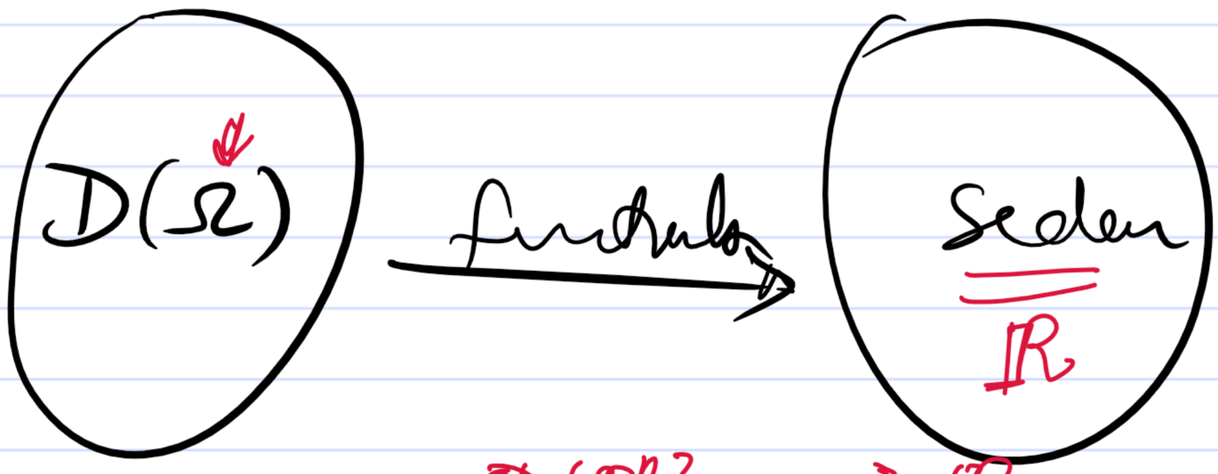
(continuous manner)



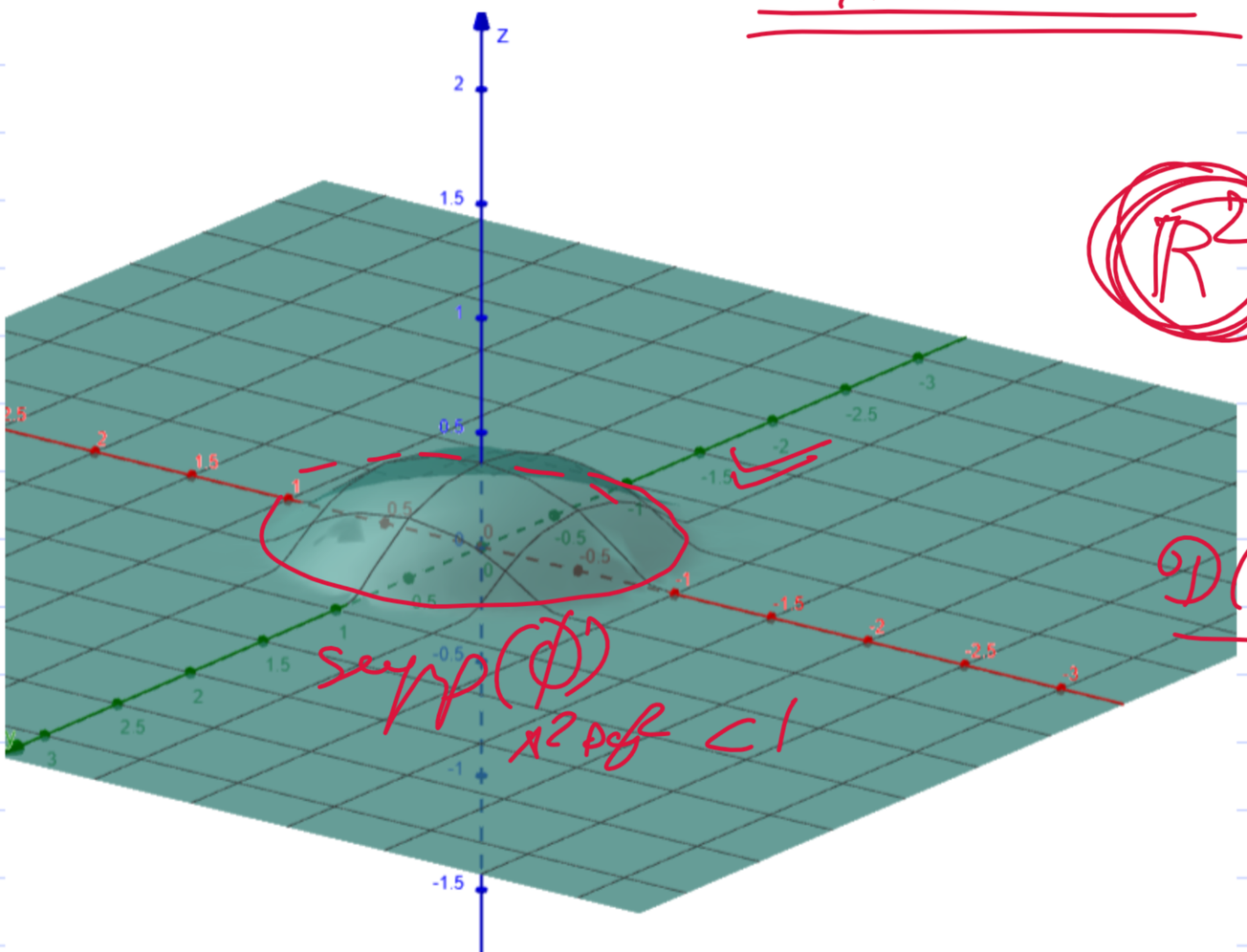
$$\mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\Phi_{(x,y)} = \begin{cases} e^{\frac{1}{x^2+y^2-1}}, & x^2+y^2<1 \\ 0, \text{ else} \end{cases}$$





$\mathcal{D}\{\mathbb{R}^n\} \longrightarrow \mathbb{R}$



"Sequential" Continuity of Linear Functionals and Distributions

Avijit

October 1, 2025

1.1 Spaces and Convergence

Let $\mathscr{D}(\mathbb{R}^n)$ denote the space of **test functions** - smooth functions with compact support.

A sequence $\{\phi_k\} \subset \mathscr{D}(\mathbb{R}^n)$ **converges to zero** in $\mathscr{D}$, denoted $\{\phi_k\} \xrightarrow{\mathscr{D}} 0$, if:

1. There exists a compact set $K \subset \mathbb{R}^n$ such that $\operatorname{supp}(\phi_k) \subset K$ for all k.
2. For every multi-index α, $D^\alpha \phi_k \to 0$ uniformly on K.

1.2 Continuous Linear Functionals

A linear functional $T : \mathscr{D}(\mathbb{R}^n) \to \mathbb{R}$ is called **continuous** if:

$$\{\phi_k\} \xrightarrow{\mathscr{D}} 0 \quad \Rightarrow \quad \{T(\phi_k)\} \to 0$$

That is, if a sequence of test functions converges to zero, then the sequence of their functionals also converges to zero. (Sequential Continuity definition)

1.3 Distributions (Generalized Functions)

A **distribution** (or **generalized function**) is a *continuous linear functional* on $\mathscr{D}(\mathbb{R}^n)$. The space of all distributions is denoted $\mathscr{D}'(\mathbb{R}^n)$.

1.4 Example: The Dirac Delta Distribution

Consider the most important example: the Dirac delta.

- **Test Function Space:** $\mathscr{D}(\mathbb{R})$
- **Linear Functional:** $\delta(\phi) = \phi(0)$ (or) $\langle \delta, \phi \rangle = \phi(0)$ Action of δ on ϕ :
- **Linearity:**

$$\begin{aligned} \delta(a\phi + b\psi) &= (a\phi + b\psi)(0) \\ &= a\phi(0) + b\psi(0) = a\delta(\phi) + b\delta(\psi) \quad \checkmark \end{aligned}$$

- **Continuity:** Let $\{\phi_k\} \xrightarrow{\mathscr{D}} 0$. By definition of convergence in $\mathscr{D}$, $\{\phi_k(0)\} \to 0$. Therefore:

$$\{\delta(\phi_k)\} = \{\phi_k(0)\} \to 0 \quad \checkmark$$

Since the Dirac delta functional is both a linear and a continuous functional, it is a **distribution**.

Continuity of Linear Functionals and Distributions

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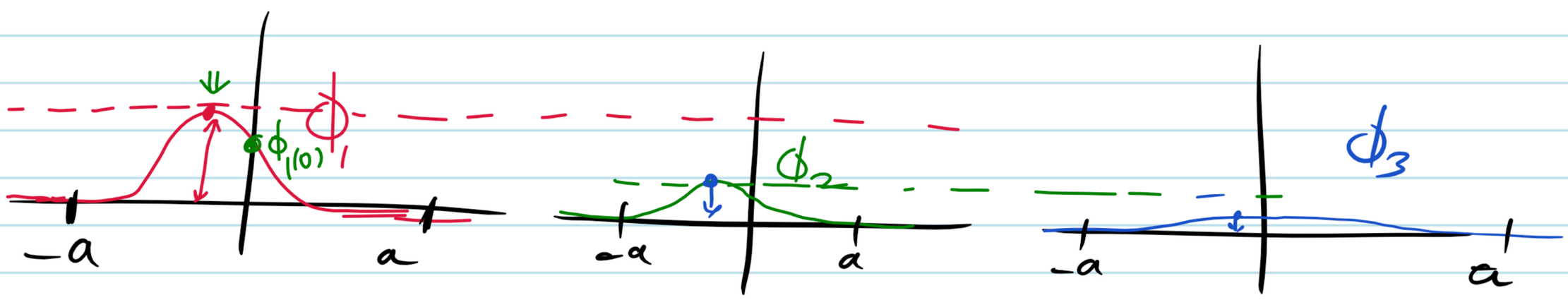
① $\delta \longrightarrow$ linear functional

② $\phi : \mathbb{R} \to \mathbb{R}$

$\mathcal{D}(\mathbb{R}) \longrightarrow \{\phi_k\} \to 0$

$\{\phi_1, \phi_2, \phi_3 \longrightarrow \cdots \} \to 0$



$n \to$ large

flat $\phi_n \to 0$

$$\lim_{n\to\infty} \left(\sup_{x\in[-a,a]} |\phi_n(x)| \right) = 0$$

$$\langle \delta, |\phi_n(x)| \rangle = |\phi_n(0)| \leq \sup_{x\in[-a,a]} |\phi_n(x)|$$

$$\Rightarrow \quad 0 \leq \langle \delta, |\phi_n(x)| \rangle \leq \sup_{x\in[-a,a]} |\phi_n(x)|$$

$$\lim_{n\to\infty} (0) \leq \lim_{n\to\infty} \langle \delta, |\phi_n(x)| \rangle \leq \lim_{n\to\infty} \left(\sup_{x\in[-a,a]} |\phi_n(x)| \right)$$

0 $\qquad$ 0 (from above)

$$\Rightarrow \lim_{n\to\infty} \langle \delta, |\phi_n(x)| \rangle = 0$$

$$\langle \delta, |\phi| \rangle = |\phi(0)| = |\langle \delta, \phi \rangle|$$

$$\lim_{n\to\infty} |\langle \delta, \phi_n(x) \rangle| = 0$$

$\{\phi_k\} \longrightarrow 0$

$\Rightarrow \lim_{n\to\infty} \langle \delta, |\phi_n(x)| \rangle = 0$

$\langle \delta, |\phi| \rangle = |\phi(0)| = |\langle \delta, \phi \rangle|$

$\lim_{n\to\infty} |\langle \delta, \phi_n(x) \rangle| = 0$

n ↑↑

$\langle \delta, \phi_n(x) \rangle \to 0$

$\{\delta[\phi_1], \delta[\phi_2], \delta[\phi_3] _ _ _ _\}$

n ↑↑ ses

↓ 0

Since δ is linear functional,

AND, $\phi_k \to 0 \Rightarrow \delta[\phi_k] \to 0$,

δ is a continuous functional as well.

$\delta \to$ linear

$\to$ Continuous

Continuity of Linear Functionals and Distributions

Avijit

October 1, 2025

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Since the Dirac delta functional is both a linear and a continuous functional, it is a **distribution**.

Continuity of the Dirac Delta Functional

Avijit

October 1, 2025

To prove that the Dirac delta functional, δ, is continuous, we must show that for any sequence of test functions $\{\phi_n\}$ converging uniformly to the zero function, the corresponding sequence of scalars $\{\langle \delta, \phi_n \rangle\}$ also converges to zero.

Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of test functions in the space of continuous functions on a compact set $[-a, a]$ containing the origin. We assume that this sequence converges uniformly to zero, which by definition means:

$$\lim_{n\to\infty} \left(\sup_{x\in[-a,a]} |\phi_n(x)| \right) = 0$$

Proof

For any term in the sequence, we apply the definition of the Dirac delta functional, $\langle \delta, \phi \rangle = \phi(0)$, to obtain the equality:

$$|\langle \delta, \phi_n \rangle| = |\phi_n(0)|$$

By the definition of the supremum, the absolute value of a function at a single point cannot exceed the supremum of its absolute values over the entire interval. This gives the inequality:

$$|\phi_n(0)| \leq \sup_{x\in[-a,a]} |\phi_n(x)|$$

Combining these two results, we establish a direct relationship between the functional's output and the norm of the input function for all n:

$$|\langle \delta, \phi_n \rangle| \leq \sup_{x\in[-a,a]} |\phi_n(x)|$$

Since it is given that the right-hand side of this inequality tends to zero as $n \to \infty$, and since $|\langle \delta, \phi_n \rangle| \geq 0$, the Squeeze Theorem implies that the left-hand side must also tend to zero. That is,

$$\lim_{n\to\infty} |\langle \delta, \phi_n \rangle| = 0$$

This demonstrates that $\{\langle \delta, \phi_n \rangle\} \to 0$, and thus the Dirac delta functional is continuous. ■